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Some New Result in Topological Space for Non-Symmetric Rational Expression Concerning Benach Space

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ABSTRACT : In the present paper some new results in topological spaces for non -symmetric relational expression conserning banach space are established.

Above results are motivated by Kirk, Singh and Chartarjee, Sharma and Rajputh, Yadava *et al.*

Keywords : Fixed point , common fixed point, Topological spaces, Banach Space.

I. INTRODUCTION

The study of Non-Contraction mapping conserning the existence of fixed point draws attention of various authors in non linear analysis dealing with the study of Non - expansive mapping and the existence of fixed points.

It is well known that the differential and integral equations that arise in the physical problems are generally non linear, therefore the fixed point methods specially " Banach contraction Principle provides a powerful tool for obtaining the solution of their equations which were very difficult to solve by any other methods.

It is also true that some qualitative properties of the solution of related equations are proved by functional analysis approach. Many authers have presented valuable results with non contraction mapping in Banach space.

II. PRELIMINARY

Before starting main result we write some definitions.

Definition 1: (Topological space) It is a set X together with τ (a collection of subsets of X) satisfying the following axioms

(i) The non empty set and set X are in τ.

(ii) τ is closed under arbitrary union.

(iii) τ is closed under finite intersection.

The collection τ is called a topology on X.

Example 1 (*a*): $X = \{1, 2, 3, 4\}$and collection $\tau =\{ \{\}, \{1, 2, 3, 4\} \}$ of only the subsets of X required by axioms form a topology, the trivial topology.

Example (*b*): $X = \{1, 2, 3, 4\}$and collection $\tau =\{ \{\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \}$of six subsets of X form another topology.

Definition 2: (Banach space) A Banach space $(X, \|.\|)$ is a normed vector space such that X is complete under the metric induced by the norm $\|.\|$.

Example 2 (*a*): The set of continuous functions on closed interval of real line with the norm $\|.\|$ of function f given by

$$\| f \| = \sup_{x\in X} | f(x) |$$

is a Banach space, where sup denotes the supremum.

Definition 3: (Normed linear space) let $\|.\|$ denotes a function from a linear space X into R that satisfies the following axioms

(i) $\forall x \in X, \| x \| \geq 0, \| x \| = 0$ iff $x = 0$

(ii) $\forall x, y \in X, \| x + y \| \leq \| x \| + \| y \|$

(iii) $\forall x \in X, \propto \in R, \| \alpha x \| = | \alpha | \| x \|$

$\|x\|$ is called the norm of x and (X, $\|..\|$) is called a Normed linear space.

Example 3 (*a*): $\langle R^n, \|.\| p\rangle, \forall x \in R^n, \| x \|_\infty = \max_{i=1}^{n} | x_i |$

Example 3 (*b*): $\langle l_p, \|.\| p\rangle \leq \infty, \forall \in l_p = \{x : x \in R^\infty, \sum_{i=1}^{\infty} | x |^p < \infty\}, \| x \| = \left(\sum_1^\infty | x_i |^p\right)^{1/p}$

Definition 4: A sequence $\{x_n\}$ in a normed space is said to be a Cauchy sequence if $\|x_n - x_m\| \to 0$, as $m, n \to \infty$ *i.e.* given $\varepsilon > 0$, there exist an integer N such that $\|x_n - x_m\| < \varepsilon$, for all m, $n > N$.for all m, $n > N$..

III. MAIN RESULT

Theorem 1:

Let F be a mapping of Banach space x into itselt. If F satisfies the following conditions:

$F^2 = I$, where I is the identity mapping

$$\|F(x)-F(y)\| \le \alpha\left[\frac{\|x-F(x)\|\|y-F(y)\|\|x-F(y)\|+\|x-y\|\|y-F(x)\|\|x-F(x)\|}{[\|x-F(x)\|+\|y-F(y)\|+\|x-y\|]^2}\right]$$
$$+\beta[\|x-F(x)\|+\|y-F(y)\|]+\gamma[\|x-F(y)\|+\|y-F(x)\|]+\delta[\|x-y\|]$$

For every $x, y \in X$, where $0 < \alpha, \beta, \gamma, \delta < 1$ and $6\alpha + 4\beta + 3\gamma + \delta < 2$ then, F has a fixed point, if $(2\gamma + \delta) < 1$ then F has a unique fixed point.

Proof:

Suppose x is a fixed point of X, taking

$$y = \frac{1}{2}(F+I)x, \quad z = F(y) \text{ and } u = 2y - z$$

We have $\|z-x\| = \|F(y)-F^2(x)\| = \|F(y)-FF(x)\|$

$$\le \alpha\left[\frac{\|y-F(y)\|\|F(x)-FF(x)\|\|y-FF(x)\|+\|y-F(x)\|\|F(x)-F(y)\|\|y-F(y)\|}{[\|y-F(y)\|+\|F(x)-FF(x)\|+\|y-F(x)\|]^2}\right]$$

$$+\beta[\|y-F(y)\|+\|F(x)-FF(x)\|]+\gamma[\|y-FF(x)\|+\|F(x)-F(y)\|]+\delta[\|y-F(x)\|]$$

$$\le \alpha\left[\frac{\|y-F(y)\|\|F(x)-x\|\|y-x\|+\|y-F(x)\|\|F(x)-F(y)\|\|y-F(y)\|}{[\|y-F(y)\|+\|F(x)-x\|+\|y-F(x)\|]^2}\right]$$

$$+\beta[\|y-F(y)\|+\|F(x)-x\|]+\gamma[\|y-x\|+\|F(x)-F(y)\|]+\delta[\|y-F(x)\|]$$

$$\le \alpha\left[\frac{\|y-F(y)\|\|F(x)-x\|\frac{1}{2}\|F(x)-x\|+\frac{1}{2}\|F(x)-x\|\frac{1}{2}\|F(x)-x\|\|y-F(y)\|}{[\|y-F(y)\|+\|x-y\|]^2}\right]$$

$$+\beta[\|y-F(y)\|+\|F(x)-x\|]+\gamma[\|y-x\|+\|F(x)-y+y-F(y)\|]+\delta[\|y-F(x)\|]$$

$$\le \alpha\left[\frac{\|y-F(y)\|\|F(x)-x\|\frac{1}{2}\|F(x)-x\|+\frac{1}{2}\|F(x)-x\|\frac{1}{2}\|F(x)-x\|\|y-F(y)\|}{[\|F(y)-x\|]^2}\right]$$

$$+\beta[\|y-F(y)\|+\|F(x)-x\|]+\gamma[\|y-x\|+\|F(x)-y\|+\|y-F(y)\|]+\frac{1}{2}\delta[\|y-F(x)\|]$$

$$\leq \alpha\left[\frac{\|y-F(y)\|\|F(x)-x\|\frac{1}{2}\|F(x)-x\|+\frac{1}{2}\|F(x)-x\|\frac{1}{2}\|F(x)-x\|\|y-F(y)\|}{\left[\frac{1}{2}\|F(x)-x\|\right]^2}\right]$$

$$+\beta\left[\|y-F(y)\|+\|F(x)-x\|\right]+\gamma\left[\frac{1}{2}\|F(x)-x\|+\frac{1}{2}\|F(x)-x\|+\|y-F(y)\|\right]+\frac{1}{2}\delta\left[\|x-F(x)\|\right]$$

$$=\alpha\left[2\|y-F(y)\|+\|y-F(y)\|\right]+\beta\left[\|y-F(y)\|+\|F(x)-x\|\right]+\gamma\left[\|F(x)-x\|+\|y-F(y)\|\right]+\delta\frac{1}{2}\left[\|x-F(x)\|\right]$$

$$=\left[3\alpha+\beta+\gamma\right]\|y-F(y)\|+\left[\beta+\gamma+\frac{1}{2}\delta\right]\|F(x)-x\|$$

Therefore $\|z-x\|\leq\left[3\alpha+\beta+\gamma\right]\|y-F(y)\|+\left[\beta+\gamma+\frac{1}{2}\delta\right]\|F(x)-x\|$

Also $\|u-x\|=\|2y-z-x\| \;=\|F(x)-F(y)\|$

$$\leq \alpha\left[\frac{\|x-F(x)\|\|y-F(y)\|\|x-F(y)\|+\|x-y\|\|y-F(x)\|\|x-F(x)\|}{\left[\|x-F(x)\|+\|y-F(y)\|+\|x-y\|\right]^2}\right]$$

$$+\beta\left[\|x-F(x)\|+\|y-F(y)\|\right]+\gamma\left[\|x-F(y)\|+\|y-F(x)\|\right]+\delta\left[\|x-y\|\right]$$

$$\leq \alpha\left[\frac{\|x-F(x)\|\|y-F(y)\|\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|\frac{1}{2}\|x-F(x)\|\|x-F(x)\|}{\left[\|F(x)-F(y)\|\right]^2}\right]$$

$$+\beta\left[\|x-F(x)\|+\|y-F(y)\|\right]+\gamma\left[\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|\right]+\frac{1}{2}\delta\left[\|x-F(x)\|\right]$$

$$\leq \alpha\left[\frac{\|x-F(x)\|\|y-F(y)\|\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|\frac{1}{2}\|x-F(x)\|\|x-F(x)\|}{\left[\frac{1}{2}\|x-F(x)\|\right]^2}\right]$$

$$+\beta\left[\|x-F(x)\|+\|y-F(y)\|\right]+\gamma\left[\|x-F(x)\|\right]+\frac{1}{2}\delta\left[\|x-F(x)\|\right]$$

$$=\left[2\alpha+\beta\right]\|y-F(y)\|+\left[\alpha+\beta+\gamma+\frac{1}{2}\delta\right]\|x-F(x)\|$$

Now

$\|z-u\|\leq\|z-x\|+\|x-u\|$

$$= [3\alpha+\beta+\gamma]\|y-F(y)\| + \left[\beta+\gamma+\frac{1}{2}\delta\right]\|F(x)-x\| + [2\alpha+\beta]\|y-F(y)\| + \left[\alpha+\beta+\gamma+\frac{1}{2}\delta\right]\|x-F(x)\|$$

$$= [5\alpha+2\beta+\gamma]\|y-F(y)\| + [\alpha+2\beta+2\gamma+\delta]\|x-F(x)\|$$

Also $\|z-u\| = \|F(y)-2y+z\| = 2\|F(y)-y\|$

$$2\|F(y)-y\| \le [5\alpha+2\beta+\gamma]\|y-F(y)\| + [\alpha+2\beta+2\gamma+\delta]\|x-F(x)\|$$

$$\Rightarrow \quad \|F(y)-y\| \le \left[\frac{5}{2}\alpha+\beta+\frac{1}{2}\gamma\right]\|y-F(y)\| + \left[\frac{1}{2}\alpha+\beta+\gamma+\frac{1}{2}\delta\right]\|x-F(x)\|$$

$$\Rightarrow \quad \|y-f(y)\| \le \left[\frac{\frac{\alpha}{2}+\beta+\gamma+\frac{\delta}{2}}{1-\frac{5}{2}\alpha-\beta-\frac{\gamma}{2}}\right]\|x-F(x)\|$$

$$\Rightarrow \quad \|y-f(y)\| \le q\|x-F(x)\| \text{ where } q = \frac{\frac{\alpha}{2}+\beta+\gamma+\frac{\delta}{2}}{1-\frac{5}{2}\alpha-\beta-\frac{\gamma}{2}} < 1$$

Since $6\alpha+4\beta+3\gamma+\delta < 2$

On taking $G = \frac{1}{2}(F+I)$ for every $x\varepsilon X$

$$\left[\|G^2(x)-G(x)\| = \|G(y)-y\| = \left\|\frac{1}{2}(F+I)y-y\right\|\right] = \frac{1}{2}\|y-F(y)\| \le q\|x-F(x)\|$$

By definition of q we claim that $\{G^n(x)\}$ is a cauchy sequence in X. Therefore by the property of compactness. $\{G^n(x)\}$ converges to some element χ_0 in X.

$$\lim_{n\to\infty} G^n(x) = \chi_0$$

$\Rightarrow$ $G(\chi_0) = \chi_0$

$\Rightarrow$ $F(\chi_0) = \chi_0$

i.e. χ_0 is a fixed point of F.

For uniqueness if possible let $y_0(\ne x_0)$ be another fixed point of F then

$$\|x_0-y_0\| = \|F(x_0)-F(y_0)\|$$

$$\le \alpha\left[\frac{\|xo-F(xo)\|\|yo-F(yo)\|\|xo-F(yo)\| + \|xo-yo\|\|yo-F(xo)\|\|xo-F(xo)\|}{[\|xo-F(xo)\|+\|yo-F(yo)\|+\|xo-yo\|]^2}\right]$$

$$+\beta[\|x_0-F(x_0)\|+\|y_0-F(y_0)\|] + \gamma[\|x_0-F(y_0)\|+\|y_0-F(x_0)\|] + \delta[\|x_0-y_0\|]$$

$$= \beta[\|y_0-F(y_0)\|] + \gamma[\|x_0-F(y_0)\|+\|y_0-F(x_0)\|] + \delta[\|x_0-y_0\|]$$

$$= \gamma\left[\|x_0 - F(y_0)\| + \|y_0 - F(x_0)\|\right] + \delta\left[\|x_0 - y_0\|\right]$$

$$= \gamma\left[\|x_0 - y_0\| + \|y_0 - x_0\|\right] + \delta\left[\|x_0 - y_0\|\right] = (2\gamma + \delta)\left[\|x_0 - y_0\|\right]$$

Since $(2\gamma + \delta) < 1$, therefore $\|x_0 = y_0\| = 0$

$\Rightarrow$ $x_0 = y_0$

This completes the proof.

Now we prove the following theorem which generalize the theorem 1.

Let K be closed and convex subset of a Banach space X. Let $F : K \to K$, $G : K \to K$ satisfy the conditions :

F and G commutes ... (1)

$F^2 = I$ and $G^2 = I$, where I is denotes the identity mapping. ... (2)

$$\|F(x) - F(y)\| \le \alpha\left[\frac{\|G(x)-F(x)\|\|G(y)-F(y)\|\|G(x)-F(y)\| + \|G(x)-G(y)\|\|G(y)-F(x)\|\|G(x)-F(x)\|}{[\|G(x)-F(x)\| + \|G(y)-F(y)\| + \|G(x)-G(y)\|]^2}\right]$$

$$+\beta\left[\|G(x)-F(x)\| + \|G(y)-F(y)\|\right] + \gamma\left[\|G(x)-F(y)\| + \|G(y)-F(x)\|\right] + \delta\left[\|G(x)-G(y)\|\right] \quad \text{... (3)}$$

For every $x, y \in X, 0 \le \alpha, \beta, \delta, \gamma$ with $6\alpha + 4\beta + 3\gamma + \delta < 2$ then there exit atleast one fixed point $x_0 \varepsilon X$ such that $F(x_0) = G(x_0) = x_0$. Further if $(2\gamma + \delta) < 1$ then x is the unique fixed point of F and G.

Proof:

From (1) and (2) it follows that $(FG)^2 = I$ and (2) and (3) implies

$$\|FGG(x) - FGG(y)\| = \|FG^2(x) - FG^2(y)\| = \alpha\left[\frac{\|GG^2(x)-FG^2(x)\|\|GG^2(y)-FG^2(y)\|\|GG^2(x)-FG^2(y)\| + \|GG^2(x)-GG^2(y)\|\|GG^2(y)-FG^2(x)\|\|GG^2(x)-FG^2(x)\|}{\left[\|GG^2(x)-FG^2(x)\| + \|GG^2(y)-FG^2(y)\| + \|GG^2(x)-GG^2(y)\|\right]^2}\right]$$

$$+\beta\left[\|GG^2(x)-FG^2(x)\| + \|GG^2(y)-FG^2(y)\|\right] + \gamma\left[\|GG^2(x)-FG^2(y)\| + \|GG^2(y)-FG^2(x)\|\right] + \delta\|GG^2(x)-GG^2(y)\|$$

$$= \alpha\left[\frac{\|G(x)-FG.G(x)\|\|G(y)-FG.G(y)\|\|G(x)-FG.G(y)\| + \|G(x)-G(y)\|\|G(y)-FG.G(x)\|\|G(x)-FG.G(x)\|}{\left[\|G(x)-FG.G(x)\| + \|G(y)-FG.G(y)\| + \|G(x)-G(y)\|\right]^2}\right]$$

$$+\beta\left[\|G(x)-FG.G(x)\| + \|G(y)-FG.G(y)\|\right] + \gamma\left[\|G(x)-FG.G(y)\| + \|G(y)-FG.G(x)\|\right] + \delta\|G(x)-G(y)\|$$

Now we put $G(x) = z$ and $G(y) = w$ then we get

$$\|FG(z)-FG(w)\| \le \left[\frac{\|z-FG(z)\|\|w-FG(w)\|\|z-FG(w)\| + \|z-w\|\|w-FG(z)\|\|z-FG(z)\|}{[\|z-FG(z)\|+\|w-FG(w)\|+\|z-w\|]^2}\right]$$

$$+\beta[\|z-FG(z)\|+\|w-FG(w)\|]+\gamma[\|z-FG(w)\|+\|w-FG(z)\|]+\delta\|z-w\|$$

We have $(FG)^2 = I$ and so by theorem 1, FG has atleast one fixed point say x_0 in K.

i.e. $F(x_0) = x_0$ ... (4)

and so $F(FG)x_0 = F(x_0)$

$G(x_0) = F(x_0)$ ... (5)

Now

$$\|F(x_0)-x_0\| = \|F(x_0)-F^2(x_0)\| = \|F(x_0)-FF(x_0)\|$$

$$\le \alpha\left[\frac{\|G(x_0)-F(x_0)\|\|GF(x_0)-FF(x_0)\|\|G(x_0)-FF(x_0)\| + \|G(x_0)-GF(x_0)\|\|GF(x_0)-F(x_0)\|\|G(x_0)-F(x_0)\|}{[\|G(x_0)-F(x_0)\|+\|GF(x_0)-FF(x_0)\|+\|G(x_0)-GF(x_0)\|]^2}\right]$$

$$+\beta[\|G(x_0)-F(x_0)\|+\|GF(x_0)-FF(x_0)\|]+\gamma[\|G(x_0)-FF(x_0)\|+\|GF(x_0)-F(x_0)\|]$$
$$+\delta\|G(x_0)-GF(x_0)\|$$

$$= \alpha\left[\frac{\|G(x_0)-F(x_0)\|\|x_0-x_0\|\|G(x_0)-x_0\| + \|G(x_0)-x_0\|\|x_0-F(x_0)\|\|G(x_0)-F(x_0)\|}{[\|G(x_0)-F(x_0)\|+\|x_0-x_0\|+\|G(x_0)-x_0\|]^2}\right]$$
$$+\beta[\|G(x_0)-F(x_0)\|+\|x_0-x_0\|]+\gamma[\|G(x_0)-x_0\|+\|x_0-F(x_0)\|]+\delta\|G(x_0)-x_0\|$$

$$= \alpha\left[\frac{\|F(x_0)-F(x_0)\|\|x_0-x_0\|\|F(x_0)-x_0\| + \|F(x_0)-x_0\|\|x_0-F(x_0)\|\|F(x_0)-F(x_0)\|}{[\|F(x_0)-F(x_0)\|+\|x_0-x_0\|+\|F(x_0)-x_0\|]^2}\right]$$
$$+\beta[\|F(x_0)-F(x_0)\|+\|x_0-x_0\|]+\gamma[\|F(x_0)-x_0\|+\|x_0-F(x_0)\|]+\delta\|F(x_0)-x_0\|$$

$$= (2\gamma+\delta)\|F(x_0)-x_0\|$$

Since $(2\gamma + \delta) < 1$ it follows that $F(x_0) = x_0$

i.e. x_0 is fixed point of F, but $F(x_0) = G(x_0)$ therefore we have $G(x_0) = x_0$

i.e. x_0 is common fixed point of F and G

Now we shall prove that x_0 is unique common fixed point of F and G. If possible, let y_0 be another fixed point of F and G.

Now from (1), (2), (3), (4) and (5), we have

$$\|x_0 - y_0\| = \|F^2(x_0) - F^2(y_0)\| = \|FF(x_0) - FF(y_0)\|$$

$$\leq \alpha \left[\frac{\|GF(x_0) - FF(x_0)\| \|GF(y_0) - FF(y_0)\| \|GF(x_0) - FF(y_0)\| + \|GF(x_0) - GF(y_0)\| \|GF(y_0) - FF(x_0)\| \|GF(x_0) - FF(x_0)\|}{[\|GF(x_0) - FF(x_0)\| + \|GF(y_0) - FF(y_0)\| + \|GF(x_0) - FF(y_0)\|]^2} \right]$$

$$+\beta\left[\|GF(x_0) - FF(x_0)\| + \|GF(y_0) - FF(y_0)\|\right] + \gamma\left[\|GF(x_0) - FF(y_0)\| + \|GF(y_0) - FF(x_0)\|\right] + \delta\|GF(x_0) - GF(y_0)\|$$

$$\leq \alpha \left[\frac{\|x_0 - x_0\| \|y_0 - y_0\| \|x_0 - y_0\| + \|x_0 - y_0\| \|y_0 - x_0\| \|x_0 - x_0\|}{[\|x_0 - x_0\| + \|y_0 - y_0\| + \|x_0 - y_0\|]^2} \right] + \beta\left[\|x_0 - x_0\| + \|y_0 - y_0\|\right] + \gamma\left[\|x_0 - y_0\| + \|y_0 - x_0\|\right] + \delta\left[\|x_0 - y_0\|\right] = (2\gamma + \delta)\|x_0 - y_0\|$$

Therefore $\|x_0 - y_0\| \leq (2\gamma + \delta)\|x_0 - y_0\|$

Since $(2\gamma + \delta) < 1$, it follows that $x_0 = y_0$, proving the uniqueness of x_0. The proof of the theorem 2 is completed.

Now we prove the following theorem which generalize theorem 1 and 2.

Theorem 3:

Let F, G and H be three mapping of Banach space X into itself such that:

$$FG = GF,\ GH = HG \text{ and } FH - HF \qquad \dots (1)$$

$$F^2 = I,\ G^2 = I,\ H^2 = I \qquad \dots (2)$$

where I is the identity mapping.

$$\|F(x) - F(y)\| \leq \alpha \left[\frac{\|GH(x) - F(x)\| \|GH(y) - F(y)\| \|GH(x) - F(y)\| + \|GH(x) - GH(y)\| \|GH(y) - F(x)\| \|GH(x) - F(x)\|}{[\|GH(x) - F(x)\| + \|GH(y) - F(y)\| + \|GH(x) - GH(y)\|]^2} \right]$$

$$= \beta\left[\|GH(x) - F(x)\| + \lfloor GH(y) - F(y) \rfloor\right] + \gamma\left[\|GH(x) - F(y)\| + \lfloor GH(y) - F(x) \rfloor\right] + \delta\left[\|GH(x) - GH(y)\|\right] \qquad \dots (3)$$

For every $x, y \in X$ and $0 \leq \alpha, \beta, \gamma, \delta < 1$ such that $6\alpha + 4\beta + 3\gamma + \delta < 2$ then, F, G and H have atleast one fixed point. Further if $(2\gamma + \delta) < 1$, then x_0 is the unique fixed point of F, G and H.

Proof:

From (1) and (2) it follows that $(FGH)^2 = I$, where I is identity mapping.

From (2) and (3) we have

$$\|FGH.G(x)-FGH.G(y)\| \le \alpha\left[\frac{\|(GH)^2G(x)-FGHG(x)\|\|(GH)^2G(y)-FGHG(y)\|\|(GH)^2G(x)-FGHG(y)\| + \|(GH)^2G(x)-(GH)^2G(y)\|\|(GH)^2G(y)-FGHG(x)\|\|(GH)^2G(x)-FGHG(x)\|}{\left[\|(GH)^2G(x)-FGHG(x)\|+\|(GH)^2G(y)-FGHG(y)\|+\|(GH)^2G(x)-(GH)^2G(y)\|\right]^2}\right]$$

$$+\beta\left[\|(GH)^2G(x)-FGHG(x)\|+\|(GH)^2G(y)-FGHG(y)\|\right]+\gamma\left[\|(GH)^2G(x)-FGHG(y)\|+\|(GH)^2G(y)-FGHG(x)\|\right]$$
$$+\delta\left[\|(GH)^2G(x)-(GH)^2G(y)\|\right]$$

$$\|FGH.G(x)-FGH.G(y)\| \le \alpha\left[\frac{\|G(x)-FGHG(x)\|\|G(y)-FGHG(y)\|\|G(x)-FGHG(y)\| + \|G(x)-G(y)\|\|G(y)-FGHG(x)\|\|G(x)-FGHG(x)\|}{\left[\|G(x)-FGHG(x)\|+\|G(y)-FGHG(y)\|+\|G(x)-G(y)\|\right]^2}\right]$$

$$+\beta\left[\|G(x)-FGHG(x)\|+\|G(y)-FGHG(y)\|\right]+\gamma\left[\|G(x)-FGHG(y)\|+\|G(y)-FGHG(x)\|\right]+\delta\left[\|G(x)-G(y)\|\right]$$

If we put $G(x) = z$ and $G(y) = w$, we get

$$\|FGH.(z)-FGH.(w)\| \le \alpha\left[\frac{\|z-FGH(z)\|\|w-FGH(w)\|\|z-FGH(w)\| + \|z-w\|\|w-FGH(z)\|\|z-FGH(z)\|}{\left[\|z-FGH(z))\|+\|w-FGH(w)\|+\|z-w\|\right]^2}\right]$$

$$+\beta\left[\|z)-FGH(z))\|+\|w-FGH(w)\|\right]+\gamma\left[\|z-FGH(w)\|+\|w-FGH(z)\|\right]+\delta\left[\|z-w\|\right]$$

If we put $FGH = N$, we get

$$\|N(z)-N(w)\| \le \alpha\left[\frac{\|z-N(z)\|\|w-N(w)\|\|z-N(w)\|+\|z-w\|\|w-N(z)\|\|z-N(z)\|}{\left[\|z-N(z)\|+\|w-N(w)\|+\|z-w\|\right]^2}\right]$$

$$+\beta\left[\|z-N(z)\|+\|w-N(w)\|\right]+\gamma\left[\|z-N(w)\|+\|w-N(z)\|\right]+\delta\left[\|z-w\|\right]$$

Let $$w=\frac{1}{2}(N+I)z$$

$$N(w) = s \text{ and } t = 2w - s \qquad \text{... (4)}$$

Now by (4) we have

$$\|s-z\| = \|N(w)-z\| = \|N(w)-N^2(z)\| = \|N(w)-NN(z)\|$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-NN(z)\|\|w-NN(z)\|+\|w-N(z)\|\|N(z)-N(w)\|\|w-N(w)\|}{[\|w-N(w)\|+\|N(z)-NN(z)\|+\|w-N(z)\|]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-NN(z)\|\right]+\gamma\left[\|w-NN(z)\|+\|N(z)-N(w)\|\right]+\delta\left[\|w-N(z)\|\right]$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-z\|\|w-z)\|+\|w-N(z)\|\|N(z)-N(w)\|\|w-N(w)\|}{[\|w-N(w)\|+\|N(z)-z\|+\|w-N(z)\|]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\|w-z\|+\|N(z)-w\|++\|w-N(w)\|\right]+\delta\left[\|w-N(z)\|\right]$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-z\|\left\|\frac{1}{2}(N+I)z-z)\right\| +\left\|\frac{1}{2}(N+I)z-N(z)\right\|\left\|N(z)-N(\frac{1}{2}(N+I)z)\right\|\|w-N(w)\|}{[\|w-N(w)\|+\|w-z\|]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\left\|\frac{1}{2}(N+I)z-z\right\|+\left\|N(z)-\frac{1}{2}(N+I)z\right\|+\|w-N(w)\|\right]$$
$$+\delta\left[\left\|\frac{1}{2}(N+I)z-N(z)\right\|\right]$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-z\|\frac{1}{2}\|N(z)-z)\|+\frac{1}{2}\|N(z)-z)\|\frac{1}{2}\|N(z)-z)\|\|w-N(w)\|}{[\|N(w)-z\|]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\frac{1}{2}\|N(z)-z)\|+\frac{1}{2}\|N(z)-z)\|+\|w-N(w)\|\right]$$
$$+\delta\left[\frac{1}{2}\|N(z)-z)\|\right]$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-z\|\frac{1}{2}\|N(z)-z)\|+\frac{1}{2}\|N(z)-z)\|\frac{1}{2}\|N(z)-z)\|\|w-N(w)\|}{\left[\left\|N(\frac{1}{2}(N+I)z-z\right\|\right]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\|N(z)-z)\|+\|w-N(w)\|\right]+\delta\left[\frac{1}{2}\|N(z)-z)\|\right]$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\|N(z)-z\|\frac{1}{2}\|N(z)-z)\|+\frac{1}{2}\|N(z)-z)\|\frac{1}{2}\|N(z)-z)\|\|w-N(w)\|}{\left[\frac{1}{2}\|N(z)-z)\|\right]^2}\right]$$

$$+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\|N(z)-z)\|+\|w-N(w)\|\right]+\delta\left[\frac{1}{2}\|N(z)-z)\|\right]$$

$$\leq \alpha\left[2\|w-N(w)\|+\|w-N(w)\|\right]+\beta\left[\|w-N(w)\|+\|N(z)-z\|\right]+\gamma\left[\|N(z)-z\|+\|w-N(w)\|\right]+\delta\frac{1}{2}\|N(z)-z\|$$

$$\leq (3\alpha+\beta+\gamma)\|w-N(w)\|+(\beta+\gamma+\frac{\delta}{2})\|N(z)-z\|$$

Therefore,

$$\|s-z\|\leq (3\alpha+\beta+\gamma)\|w-N(w)\|+(\beta+\gamma+\frac{\delta}{2})\|N(z)-z\|$$

Now $\|\text{t-z}\|\leq\|2w-s-z\|=\left\|2(\frac{1}{2}(N+I)z-s-z\right\|=\|N(z)+z-s-z\|=\|N(z)-s\|=\|N(z)-N(w)\|$

$$\leq \alpha\left[\frac{\|z-N(z)\|\|w-N(w)\|\|z-N(w)\|+\|z-w\|\|w-N(z)\|\|z-N(z)\|}{\left[\|z-N(z)\|+\|w-N(w)\|+\|z-w\|\right]^2}\right]$$

$$+\beta\left[\|z-N(z)\|+\|w-N(w)\|\right]+\gamma\left[\|z-N(w)\|+\|w-N(z)\|\right]+\delta\|z-w\|$$

$$\leq \alpha\left[\frac{\|z-N(z)\|\|w-N(w)\|\left\|z-N(\frac{1}{2}(N+I)z)\right\| +\left\|z-(\frac{1}{2}(N+I)z\right\|\left\|(\frac{1}{2}(N+I)z-N(z)\right\|\|z-N(z)\|}{\left[\|N(z)-N(w)\|\right]^2}\right]$$

$$+\beta\left[\|z-N(z)\|+\|w-N(w)\|\right]+\gamma\left[\left\|z-N(\frac{1}{2}(N+I)z\right\|+\left\|\frac{1}{2}(N+I)z-N(z)\right\|\right]+\delta\left\|z-\frac{1}{2}(N+I)z\right\|$$

$$\leq \alpha \left[\frac{\|z-N(z)\|\|w-N(w)\|\frac{1}{2}\|z-N(z)\|+\frac{1}{2}\|z-N(z)\|\frac{1}{2}\|z-N(z)\|\|z-N(z)\|}{\left[\left\|\frac{1}{2}\|z-N(z)\|\right\|\right]^2} \right]$$

$$+\beta\left[\|z-N(z)\|+\|w-N(w)\|\right]+\gamma\left[\frac{1}{2}\|z-N(z)\|+\frac{1}{2}\|z-N(z)\|\right]+\delta\frac{1}{2}\|z-N(z)\|$$

$$=\alpha\left[2\|w-N(w)\|+\|z-N(z)\|\right]+\beta\left[\|w-N(w)\|+\|z-N(z)\|\right]+\gamma\|z-N(z)\|+\delta\frac{1}{2}\|z-N(z)\|$$

$$=(2\alpha+\beta)\|w-N(w)\|+(\alpha+\beta+\gamma+\frac{1}{2}\delta)\|z-N(z)\| \qquad \ldots (6)$$

Now $\|s-t\| \leq \|s-z\|+\|z-t\|$

Therefore from (5) and (6) we write

$$\|s-t\| \leq (3\alpha+\beta+\gamma)\|w-N(w)\|+(\beta+\gamma+\frac{\delta}{2})\|N(z)-z\|$$

$$+(2\alpha+\beta)\|w-N(w)\|+(\alpha+\beta+\gamma+\frac{1}{2}\delta)\|z-N(z)\| \qquad \ldots (7)$$

$$\leq (5\alpha+2\beta+\gamma)\|w-N(w)\|+(\alpha+2\beta+2\gamma+\delta)\|z-N(z)\|$$

Also, $\|s-t\| = \|N(w)-(2w-s)\| = \|N(w)-2w+N(w)\| = 2\|N(w)-w\|$

Putting all these values in in equality (7) we get,

$$2\|N(w)-w\| \leq 5\alpha+2\beta+\gamma)\|w-N(w)\|+(\alpha+2\beta+2\gamma+\delta)\|z-N(z)\|$$

$\Rightarrow$ $\|N(w)-w\| \leq (\frac{5}{2}\alpha+\beta+\frac{\gamma}{2})\|w-N(w)\|+(\frac{1}{2}\alpha+\beta+\gamma+\frac{1}{2}\delta)\|z-N(z)\|$

$\Rightarrow$ $\|N(w)-w\| \leq q\|z-N(z)\|$

where,

$$q=\frac{\frac{1}{2}\alpha+\beta+\gamma+\frac{1}{2}\delta}{1-\frac{5}{2}\alpha-\beta-\frac{\gamma}{2}}<1$$

Since $5\alpha+4\beta+3\gamma+\delta<2$

On taking $G=\frac{1}{2}(N+I)$ then for any $z \varepsilon X$

$$\|G^2(z)-G(z)\| = \|G(w)-w\| = \left\|\frac{1}{2}(N+I)w-w\right\| = \frac{1}{2}\|N(w)-w\| \leq \frac{q}{2}\|z-N(z)\|$$

By definition of q we claim that $\{G^n(x)\}$ is a Cauchy sequence in X. By compactness, $\{G^n(x)\}$ converges to some element x_0 in X.

i.e. $\lim_{n\to\infty} G^n(x_0) = x_0$

$=>> G(x_0) = x_0$

Hence $N(x_0) = x_0$

or $FGH(x_0) = x_0$ because $N = FGH$ ... (8)

And so $GH(FGH)(x_0) = GH(x)$

or $F(x_0) = GH(x_0)$ ... (9)

Also $H(FGH)(x_0) = (x_0)$

$FG(x_0) = H(x_0)$ ... (10)

Now using (1), (2), (3) and (8), (9) and (10) we have

$$\|H(x_0)-x_0\| = \|FG(x_0)-F^2(x_0)\| = \|FG(x_0)-F(F(x_0))\|$$

$$\le \alpha\left[\frac{\|GHG(x_0)-FG(x_0)\|\|GHF(x_0)-FF(x_0)\|\|GHG(x_0)-FF(x_0)\| + \|GHG(x_0)-GHF(x_0)\|\|GHF(x_0)-FG(x_0)\|\|GHG(x_0)-FG(x_0)\|}{\left[\|GHG(x_0)-FG(x_0)\|+\|GHF(x_0)-FF(x_0)\|+\|GHG(x_0)-GHF(x_0)\|\right]^2}\right]$$
$$+\beta\left[\|GHG(x_0)-FG(x_0)\|+\|GHF(x_0)-FF(x_0)\|\right]+\gamma\left[\|GHG(x_0)-FF(x_0)\|+\|GHF(x_0)-FG(x_0)\|\right]$$
$$+\delta\left\|\left[\|GHG(x_0)-GHF(x_0)\|\right]\right\|$$

$$\le \alpha\left[\frac{\|H(x_0)-H(x_0)\|\|x_0-x_0\|\|H(x_0)-x_0\| + \|H(x_0)-x_0\|\|x_0-H(x_0)\|\|H(x_0)-H(x_0)\|}{\left[\|H(x_0)-H(x_0)\|+\|x_0-x_0\|+\|H(x_0)-x_0\|\right]^2}\right]$$
$$+\beta\left[\|H(x_0)-H(x_0)\|+\|x_0-x_0\|\right]+\gamma\left[\|H(x_0)-x_0\|+\|x_0-H(x_0)\|\right]+\delta\|H(x_0)-x_0\|$$

$$=(2\gamma+\delta)\|H(x_0)-x_0\|$$

Hence $\|H(x_0)-x_0\| \le (2\gamma+\delta)\|H(x_0)-x_0\|$

Since $(2\gamma+\delta)<1$, it follows that $H(x_0) = x_0$ *i.e.* x_0 is the fixed point of H.

Thus we have from (9), $G(x_0) = F(x_0)$

Again $\|F(x_0)-x_0\| = \|F(x_0)-F^2(x_0)\| = \|F(x_0)-FF(x_0)\|$

$$1 \le \alpha\left[\frac{\|GH(x_0)-F(x_0)\|\|GHF(x_0)-FF(x_0)\|\|GH(x_0)-FF(x_0)\| + \|GH(x_0)-GHF(x_0)\|\|GHF(x_0)-F(x_0)\|\|GH(x_0)-F(x_0)\|}{\left[\|GH(x_0)-F(x_0)\|+\|GH(x_0)-FF(x_0)\|+\|GH(x_0)-GHF(x_0)\|\right]^2}\right]$$

$$+\beta\left[\|GH(x_0)-F(x_0)\|+\|GHF(x_0)-FF(x_0)\|\right]+\gamma\left[\|GH(x_0)-FF(x_0)\|+\|GHF(x_0)-F(x_0)\|\right]$$
$$+\delta\left[\|GH(x_0)-GHF(x_0)\|\right]$$

$$\alpha\left[\frac{\|F(x_0)-F(x_0)\|\|x_0-x_0\|\|F(x_0)-x_0\| + \|F(x_0)-x_0\|\|x_0-F(x_0)\|\|F(x_0)-F(x_0)\|}{\left[\|F(x_0)-F(x_0)\|+\|x_0-x_0\|+\|F(x_0)-x_0\|\right]^2}\right]$$

$$+\beta\left[\|F(x_0)-F(x_0)\|+\|x_0-x_0\|\right]+\gamma\left[\|F(x_0)-x_0\|+\|x_0-F(x_0)\|\right]+\delta\left[\|F(x_0)-x_0\|\right]$$

$$=(2\gamma+\delta)\|F(x_0)-x_0\|$$

$$\|F(x_0)-x_0\|\le(2\gamma+\delta)\|F(x_0)-x_0\|$$

which is contradiction, since $(2\gamma + \delta) < 1$. Hence it follows that:

$F(x_0) = x_0$

But $F(x_0) = G(x_0)$

Therefore $F(x_0) = G(x_0) = x_0$

i.e. x_0 is the common fixed point of F, G and H. Now to conform the uniqueness of x_0, let y_0 be another common fixed point of F, G and H using (1), (2), (3) and (8), (9), (10) we get,

$$\|x_0-y_0\|=\|F^2(x_0)-F^2(y_0)\|=\|FF(x_0)-FF(y_0)\|=\|F(F(x_0))-F(F(y_0))\|$$

$$\le\alpha\left[\frac{\|GHF(x_0)-FF(x_0)\|\|GHF(y_0)-FF(y_0)\|\|GHF(x_0)-FF(y_0)\| + \|GHF(x_0)-GHF(y_0)\|\|GHF(y_0)-FF(x_0)\|\|GHF(x_0)-FF(x_0)\|}{\left[\|GHF(x_0)-FF(x_0)\|+\|GHF(y_0)-FF(y_0)\|+\|GHF(x_0)-GHF(y_0)\|\right]^2}\right]$$

$$+\beta\left[\|GHF(x_0)-FF(x_0)\|+\|GHF(y_0)-FF(y_0)\|\right]+\gamma\left[\|GHF(x_0)-FF(y_0)\|+\|GHF(y_0)-FF(x_0)\|\right]$$
$$+\delta\left[\|GHF(x_0)-GHF(y_0)\|\right]$$

$$\le\alpha\left[\frac{\|x_0-x_0\|\|y_0-y_0\|\|x_0-y_0\| + \|x_0-y_0\|\|y_0-x_0\|\|x_0-x_0\|}{\left[\|x_0-x_0\|+\|y_0-y_0\|+\|x_0-y_0\|\right]^2}\right]$$

$$+\beta\left[\|x_0-x_0\|+\|y_0-y_0\|\right]+\gamma\left[\|x_0-y_0\|+\|y_0-x_0\|\right]+\delta\left[\|x_0-y_0\|\right]=(2\gamma+\delta)\left[\|x_0-y_0\|\right]$$

Therefore $\|x_0-y_0\|\le(2\gamma+\delta)\|x_0-y_0\|$

which is contradiction, since $(2\gamma + \delta) < 1$.

Hence it follows that $x_0 = y_0$, proving the uniqueness of x_0.

This completes the proof of the theorem 3.

REFRENCES

[1] Landau H.G., On dominance relations and the structure of animal societies: III. *The condition for a score structure, Bull. Math. Biophs* **15**(1953), 143-148.

[1] Ahmad, A. and Shakil, M. Some fixed point theorem in banach space, *Non linear Funct. Anal and appl.*, **11**(2006), 343-349.

[2] Banach S., surles operation dans les ensembles abstraits et leur application aux equations integrals, *Fund. Math.*, **3**(1922), 133-181.

[3] Badshah, V.H. and Gupta, O.P, *Fixed point theorem in Banach space and 2- Banach space, Jnanabha*, **35**(2005), 73-78.

[4] Brouwder, F.E. "Non-expansive non-linear operators in Banach spaces" *Proc Nat. Acad. Sci. U.S.A.* **54**(1965) 1041-1044.

[5] Datson, W.G .Jr., fixed point of quasi non -expansive mapping, j. *Austral Math. Soc.*, **13**(1972)167-172.

[6] Yadava, R.N., Rajput, S.S., Choudhary, S. and Bhardwaj, R.K. "Some fixed point and common fixed point theorem for non contraction mapping on 2- banach spaces" *Acta Ciencia Indica* **33**(2007) 737-744.

[7] Gahlar, S. "2- Metrche raume and ihre topologiscche structure" *Math. Nadh.* **26**(1963-64) 115-148.

[8] Gohde, D. zum prinzip dev kontrativen abbilduing, *Math. Nachr*, **30**(1965),251-258.

[9] Goebel. K, An elementary proof of fixed point theorem of Browder and Kirk, Meichigan Math, j. **16**(1969), 381-383.

[10] Goebel, K and Zlotkiewics, E., *Some fixed point theorem in Banach Spaces, colloq, math.*, **23**(1971),103-106

[11] Goebel, K, Kirk, W.A. and shimt, T.N. A fixed point theorem in uniformly convex spaces, Boll, Un. Math, Italy, **4**(1973). 67-75

[12] Isekey, K., fixed point theorem in Banach Spaces, Math Sem. *Notes Kobe University*, **2**(1974), 111-115.

[13] Jong S.J., *Viscocity approximation methods for a family of finite non expansive in Banach Spaces,non linear analysis*, **64**(2006), 2536-2552.

[14] Khan, M.S., fixed point and their approximations in Banach Spaces for certain commuting mappings, *Glasgow Math. Jour.*, **23**(1982), 1-6.

[15] Khan MS and Imdad , M., fixed points of certain involutions in Banach Spaces, *J. Austral. Math. soc.*, **37**(1984), 169-177.

[16] Kirk., W.A. fixed point theorem mappings donot increase distance, Amer. *Math. Monthly*, **72**(1965) 1004-1006.

[17] Kirk A., fixed point theorem for non expansive mappings. *Contem Math*. **18**(1983) 121-140.

[18] Kirk., W.A. fixed point theorem for non-expansive mappings, lecturer notes in M*th*., springer-Verlag, *Berlim and New York*, **886**(1981) 111-120.

[19] Pathak, H.K. and Maity, A.R., *A fixed point theorem in Banach space, Acta Cleneia Indica*, 17(1991), 137-139.

[20] Rajput, S.S. and Narolia, N., *Fixed point theorem in Banach space, Acta Cleneia Indica* **17**(1991) 469-474.

[21] Qureshi, N.A. and Singh, B., *A fixed point theorem in Banach space, Acta Cleneia Indica* **11**(1995) 282-284.

[22] Sharma P.L. and Rajput S.S., *Fixed point theorem in Banach space, vikram mathematical Journal* **4**(1983) 35-38.

[23] Singh M.R. and Chatterjee, A.K., fixed point in Banach space, pure Math. Manuscript, **6**(1987) 53-61.

[24] Sharma, S and Bhagwan, A. a common fixed point on normed space, *Acta Cleneia Indica*, **31**(2003) 20-24.

[25] Verma B.P. *Application of Banach fixed point theorem to solve non linear equations and its generalization, Jnanabha*, **36**(2006) 21-23.

[26] Yadava R.N., Rajput ,S S.and Bhardwaj, R.K. " some fixed point and common fixed point theorems in Banach Spaces" *Acta Ciencia Indica* **33** No. 2 (2007).